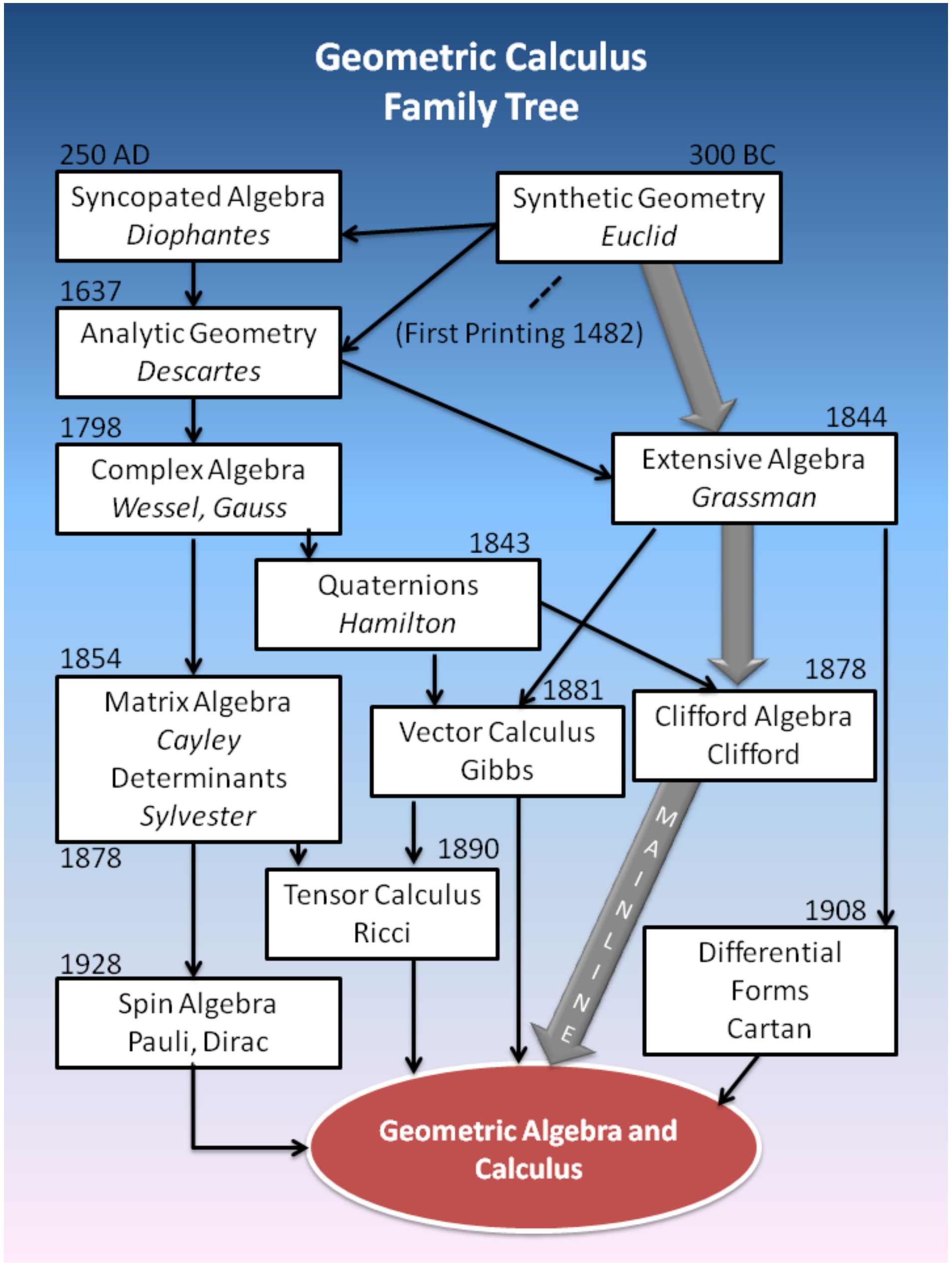
Méth. Mat. Phys. - Chapitre 12

Algèbre géométrique

EPFL


Geometric Calculus
Family Tree
250 AD
Syncopated Algebra
Diophantes
300 BC
Synthetic Geometry
Euclid
1637
Analytic Geometry
Descartes
(First Printing 1482)
1798
Complex Algebra
Wessel, Gauss
1844
Extensive Algebra
Grassman
1843
Quaternions
Hamilton
1854
Matrix Algebra
Cayley
Determinants
Sylvester
1881
Vector Calculus
Gibbs
1878
Clifford Algebra
Clifford
MAINLINE
1878
1890
Tensor Calculus
Ricci
1908
Differential
Forms
Cartan
1928
Spin Algebra
Pauli, Dirac
Geometric Algebra and
Calculus

- **Produit géométrique :** vecteurs $\boldsymbol{u}$ et $\boldsymbol{v}$

$$\tag{12.1}$$

- **Produit intérieur :** symétrique (scalaire) produit scalaire

$$\tag{12.2}$$

- **Produit extérieur :** antisymétrique (bivecteur) dual du produit vectoriel

$$\tag{12.3}$$

Le produit extérieur est associatif et non le produit vectoriel.

- **Base orthonormée :** espace en 3D : $\{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\}$

$$\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j + \hat{\boldsymbol{e}}_i \wedge \hat{\boldsymbol{e}}_j \tag{12.4}$$

$$\hat{\boldsymbol{e}}_i^2 = \hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_i = \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_i = 1 \qquad \text{et} \qquad \hat{\boldsymbol{e}}_i \wedge \hat{\boldsymbol{e}}_i = \boldsymbol{0} \tag{12.5}$$

$$(\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j)^2 = \hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j = -\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j\,\hat{\boldsymbol{e}}_j = -\,1 \qquad \text{si} \qquad \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = 0 \tag{12.6}$$

- **Algèbre géométrique :** $\mathbb{G}^3$: 8 éléments de base (2^3)

  1. 1 scalaire : 0 D

     1

  2. 3 vecteurs : 1 D

     $\hat{\boldsymbol{e}}_1, \ \hat{\boldsymbol{e}}_2, \ \hat{\boldsymbol{e}}_3$

  3. 3 bivecteurs : 2 D

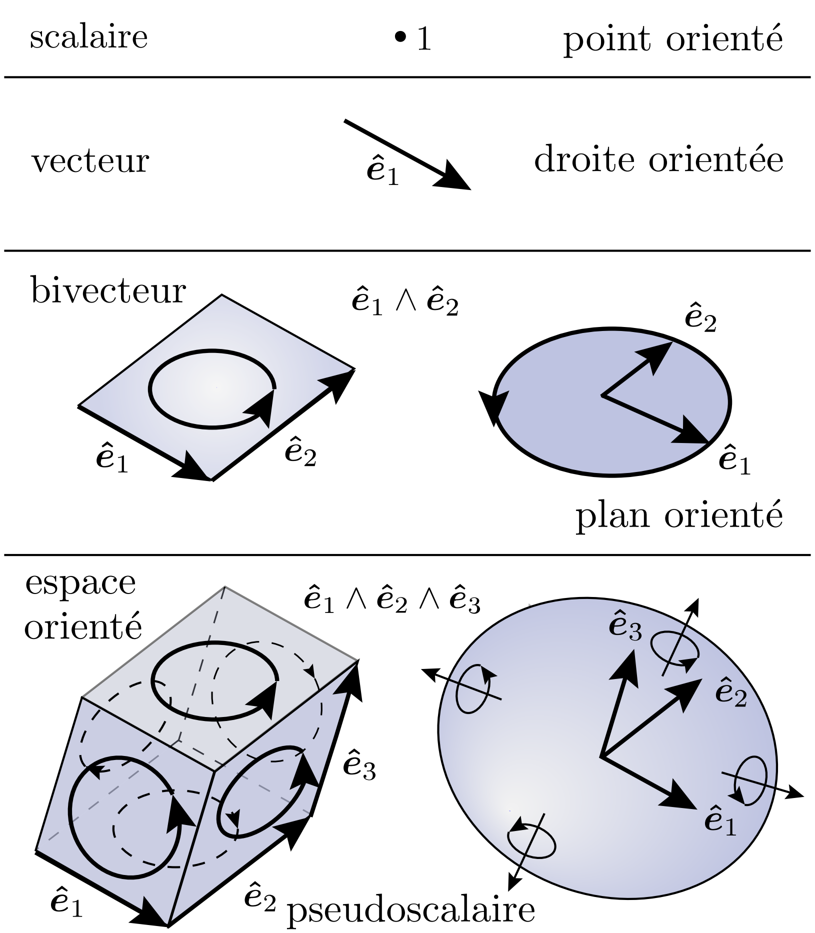
     $\hat{\boldsymbol{e}}_1 \wedge \hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{e}}_1 \, \hat{\boldsymbol{e}}_2$

     $\hat{\boldsymbol{e}}_2 \wedge \hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{e}}_2 \, \hat{\boldsymbol{e}}_3$

     $\hat{\boldsymbol{e}}_3 \wedge \hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{e}}_3 \, \hat{\boldsymbol{e}}_1$

  4. 1 pseudoscalaire : 3 D

     $I = \hat{\boldsymbol{e}}_1 \wedge \hat{\boldsymbol{e}}_2 \wedge \hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{e}}_1 \, \hat{\boldsymbol{e}}_2 \, \hat{\boldsymbol{e}}_3$



- **Pseudoscalaire :** isomorphe au nombre imaginaire i

$$ \tag{12.7} $$

- **Algèbre géométrique :** $\mathbb{G}^3$: base : 8 éléments

$$\{\, 1, \hat{\boldsymbol{e}}_1, \, \hat{\boldsymbol{e}}_2, \, \hat{\boldsymbol{e}}_3, \, \hat{\boldsymbol{e}}_1 \wedge \hat{\boldsymbol{e}}_2, \, \hat{\boldsymbol{e}}_2 \wedge \hat{\boldsymbol{e}}_3, \, \hat{\boldsymbol{e}}_3 \wedge \hat{\boldsymbol{e}}_1, \, \hat{\boldsymbol{e}}_1 \wedge \hat{\boldsymbol{e}}_2 \wedge \hat{\boldsymbol{e}}_3 \,\}$$

- **Vecteur :** minuscule (gras)

$$\boldsymbol{v} = v_1 \, \hat{\boldsymbol{e}}_1 + v_2 \, \hat{\boldsymbol{e}}_2 + v_3 \, \hat{\boldsymbol{e}}_3 \tag{12.8}$$

- **Bivecteur :** majuscule (gras)

$$\boldsymbol{B} = B_{12} \, \hat{\boldsymbol{e}}_1 \wedge \hat{\boldsymbol{e}}_2 + B_{23} \, \hat{\boldsymbol{e}}_2 \wedge \hat{\boldsymbol{e}}_3 + B_{31} \, \hat{\boldsymbol{e}}_3 \wedge \hat{\boldsymbol{e}}_1 \tag{12.9}$$

- **Trivecteur :** majuscule (normal)

$$T = T_{123} \, \hat{\boldsymbol{e}}_1 \wedge \hat{\boldsymbol{e}}_2 \wedge \hat{\boldsymbol{e}}_3 \tag{12.10}$$

- **Multivecteur :** combinaison linéaire d'éléments $\mathbb{G}^3$

$$\tag{12.11}$$

- **Produit géométrique de multivecteurs :** $\mathbb{G}^3 \times \mathbb{G}^3 \rightarrow \mathbb{G}^3$

$$(M_1 + M_2)\,(M_3 + M_4) = M_1\,M_3 + M_1\,M_4 + M_2\,M_3 + M_2\,M_4 \qquad (12.12)$$

- **Produit intérieur :** antisymétrique : vecteur et bivecteur (exercice)

(12.22)

- **Produit extérieur :** symétrique : vecteur et bivecteur (exercice)

(12.19)

- **Pseudoscalaire :** relations de commutation (exercice)

$$\boldsymbol{v}\,I = I\,\boldsymbol{v} \qquad (12.25)$$

$$\boldsymbol{B}\,I = I\,\boldsymbol{B} \qquad (12.28)$$

$$T\,I = I\,T \qquad (12.29)$$

- **Renversement :** ordre des vecteurs dans le produit (exercice)

$$s^{\dagger} = s \qquad \text{et} \qquad \boldsymbol{v}^{\dagger} = \boldsymbol{v} \tag{12.30}$$

$$\boldsymbol{B}^{\dagger} = -\,\boldsymbol{B} \qquad \text{et} \qquad T^{\dagger} = -\,T \tag{12.31}$$

- **Modules :** généralisation de la norme (exercice)

$$|\boldsymbol{v}|^2 = \boldsymbol{v}^{\dagger} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{v} = \boldsymbol{v}^2 > 0$$

$$|\boldsymbol{B}|^2 = \boldsymbol{B}^{\dagger} \cdot \boldsymbol{B} = -\,\boldsymbol{B} \cdot \boldsymbol{B} = -\,\boldsymbol{B}^2 > 0 \tag{12.32}$$

$$|T|^2 = T^{\dagger}\,T = -\,T^2 > 0$$

- **Inverses :**

(12.33)

- **Dualité :** 3D : dimensions $1 \leftrightarrow 2$ et $0 \leftrightarrow 3$

$$(12.34)$$

$$(12.37)$$

- **Dualité du dual :**

$$(\boldsymbol{v}^*)^* = -\,\boldsymbol{v} \qquad \text{et} \qquad (\boldsymbol{B}^*)^* = -\,\boldsymbol{B} \tag{12.36}$$

$$(s^*)^* = -\,s \qquad \text{et} \qquad (T^*)^* = -\,T \tag{12.39}$$

- **Identités duales de vecteurs :** (exercice)

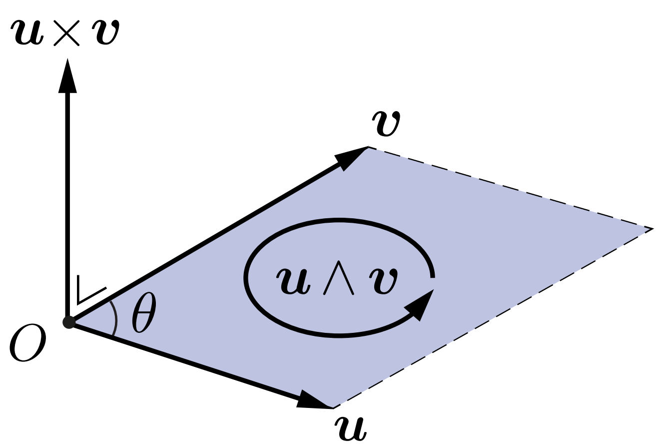
$$(12.43)$$

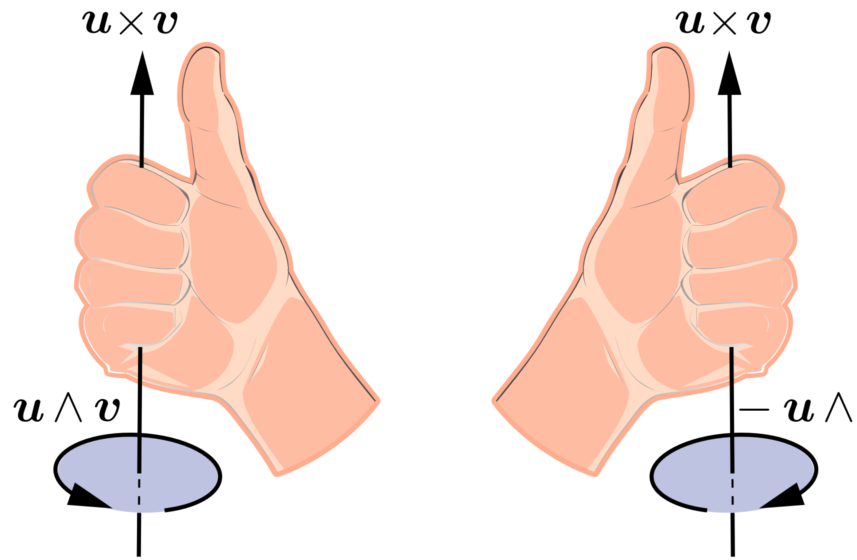
$$(12.46)$$

- **Identités duales de vecteurs et bivecteurs :** (exercice)

$$(12.49)$$

$$(12.52)$$





- **Module :** produit extérieur $\wedge$ et produit vectoriel $\times$ (exercice)

$$|\boldsymbol{u}\wedge\boldsymbol{v}| = |\boldsymbol{u}\times\boldsymbol{v}| = |\boldsymbol{u}|\,|\boldsymbol{v}|\,|\sin\theta\,| \tag{12.60}$$

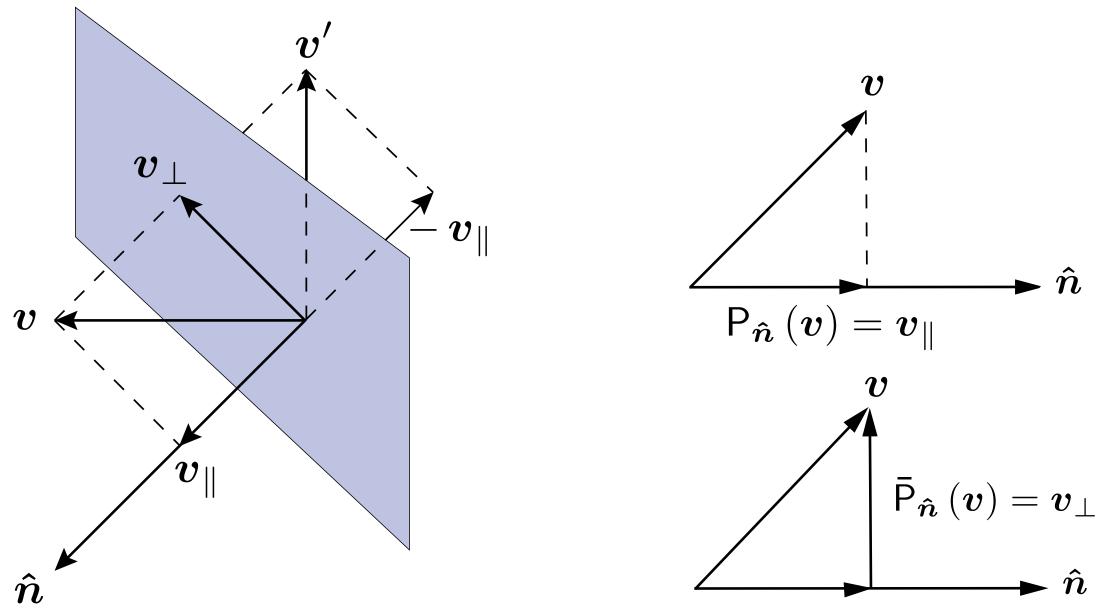
- **Dualité :** bivecteur $\rightarrow$ pseudovecteur : règle de la main droite
  pseudovecteur $\rightarrow$ bivecteur : règle de la main gauche

(12.63)

(12.65)

- **Dualité :** multivecteur

$$(M^*)^* = -\,M \tag{12.40}$$



- **Décomposition parallèle et orthogonale :** vecteur orth. au plan $\hat{\boldsymbol{n}}$

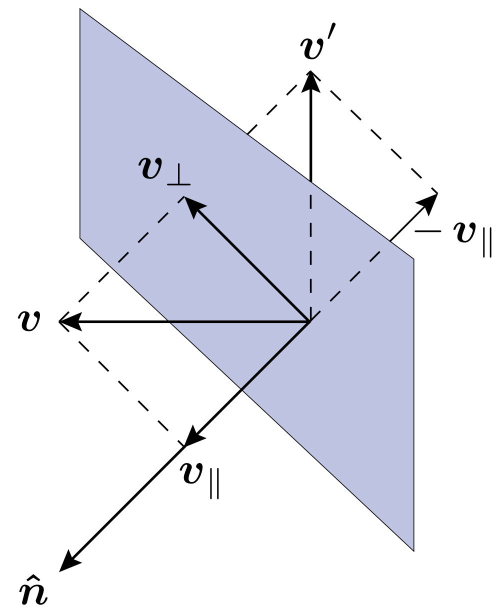
$$\tag{12.67}$$

- **Projection :** sur $\hat{\boldsymbol{n}} = \hat{\boldsymbol{n}}^{-1}$

$$\boldsymbol{v}_{\parallel} = \mathsf{P}_{\hat{\boldsymbol{n}}}\left(\boldsymbol{v}\right) = \left(\hat{\boldsymbol{n}} \cdot \boldsymbol{v}\right)\hat{\boldsymbol{n}} = \hat{\boldsymbol{n}}^{-1}\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{v}\right) \tag{12.69}$$

- **Rejection :** orthogonale à $\hat{\boldsymbol{n}}$ avec $\hat{\boldsymbol{n}} \wedge \hat{\boldsymbol{n}} = 0$ et $\hat{\boldsymbol{n}}^2 = 1$

$$\boldsymbol{v}_{\perp} = \bar{\mathsf{P}}_{\hat{\boldsymbol{n}}}\left(\boldsymbol{v}\right) = \hat{\boldsymbol{n}} \cdot \left(\hat{\boldsymbol{n}} \wedge \boldsymbol{v}\right) = \hat{\boldsymbol{n}}\left(\hat{\boldsymbol{n}} \wedge \boldsymbol{v}\right) = \hat{\boldsymbol{n}}^{-1}\left(\hat{\boldsymbol{n}} \wedge \boldsymbol{v}\right) \tag{12.70}$$



- **Réflexion :** de $\boldsymbol{v}$ selon le plan orthogonal à $\hat{\boldsymbol{n}}$

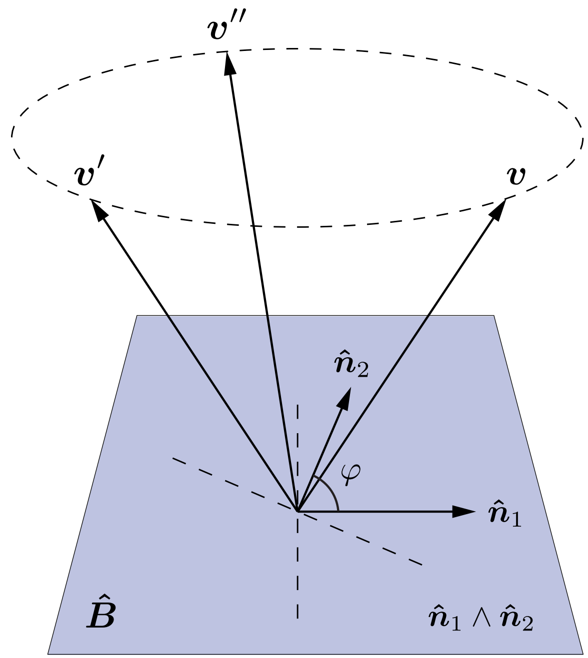
$$\boldsymbol{v} \;\mapsto\; \boldsymbol{v}' = -\,\boldsymbol{v}_\parallel + \boldsymbol{v}_\perp \tag{12.74}$$

- **Développement algébrique :** (12.22)

$$\begin{aligned} \boldsymbol{v}' &= -\,\hat{\boldsymbol{n}}\,(\hat{\boldsymbol{n}}\cdot\boldsymbol{v}) + \hat{\boldsymbol{n}}\cdot(\hat{\boldsymbol{n}}\wedge\boldsymbol{v}) = -\,(\hat{\boldsymbol{n}}\cdot\boldsymbol{v})\,\hat{\boldsymbol{n}} - (\hat{\boldsymbol{n}}\wedge\boldsymbol{v})\cdot\hat{\boldsymbol{n}} \\ &= -\,(\hat{\boldsymbol{n}}\cdot\boldsymbol{v} + \hat{\boldsymbol{n}}\wedge\boldsymbol{v})\,\hat{\boldsymbol{n}} = -\,\hat{\boldsymbol{n}}\,\boldsymbol{v}\,\hat{\boldsymbol{n}} \end{aligned} \tag{12.77}$$

- **Réflexion :** application linéaire $\boldsymbol{v} \;\mapsto\; \boldsymbol{v}'$

$$\tag{12.78}$$



- **Rotation :** dans le plan $\hat{\boldsymbol{n}}_1 \wedge \hat{\boldsymbol{n}}_2$ de bivecteur unitaire $\hat{\boldsymbol{B}}$

$$\begin{aligned} \boldsymbol{v}'' &= \mathsf{F}_{\hat{\boldsymbol{n}}_2} \circ \mathsf{F}_{\hat{\boldsymbol{n}}_1}(\boldsymbol{v}) = \mathsf{F}_{\hat{\boldsymbol{n}}_2}\Big(\mathsf{F}_{\hat{\boldsymbol{n}}_1}(\boldsymbol{v})\Big) = \mathsf{F}_{\hat{\boldsymbol{n}}_2}(\boldsymbol{v}') \\ &= -\,\mathsf{F}_{\hat{\boldsymbol{n}}_2}(\hat{\boldsymbol{n}}_1\,\boldsymbol{v}\,\hat{\boldsymbol{n}}_1) = \hat{\boldsymbol{n}}_2\,\hat{\boldsymbol{n}}_1\,\boldsymbol{v}\;\hat{\boldsymbol{n}}_1\,\hat{\boldsymbol{n}}_2 \end{aligned} \tag{12.79}$$

- **Rotor :** et renversement

$$\tag{12.80}$$

- **Rotor :** rotation : isométrie

$$|R|^2 = R^\dagger R = \hat{\boldsymbol{n}}_1\, \hat{\boldsymbol{n}}_2\, \hat{\boldsymbol{n}}_2\, \hat{\boldsymbol{n}}_1 = 1$$

$$|R^\dagger|^2 = R\, R^\dagger = \hat{\boldsymbol{n}}_2\, \hat{\boldsymbol{n}}_1\, \hat{\boldsymbol{n}}_1\, \hat{\boldsymbol{n}}_2 = 1 \qquad \text{ainsi} \qquad R^\dagger = R^{-1} \tag{12.82}$$

- **Vecteurs unitaires :** produit intérieur et produit extérieur

$$\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{n}}_2 = \cos\varphi$$

$$\hat{\boldsymbol{n}}_1 \wedge \hat{\boldsymbol{n}}_2 = \sin\varphi\, \hat{\boldsymbol{B}} \tag{12.84}$$

- **Rotor :** et renversement (12.30)

$$R = \hat{\boldsymbol{n}}_2\, \hat{\boldsymbol{n}}_1 = \hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{n}}_2 - \hat{\boldsymbol{n}}_1 \wedge \hat{\boldsymbol{n}}_2 = \cos\varphi - \sin\varphi\, \hat{\boldsymbol{B}}$$

$$R^\dagger = \hat{\boldsymbol{n}}_1\, \hat{\boldsymbol{n}}_2 = \hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{n}}_2 + \hat{\boldsymbol{n}}_1 \wedge \hat{\boldsymbol{n}}_2 = \cos\varphi + \sin\varphi\, \hat{\boldsymbol{B}} \tag{12.85}$$

- **Bivecteur :** unitaire

$$\hat{\boldsymbol{B}}^2 = \hat{\boldsymbol{B}}\, \hat{\boldsymbol{B}} = -\,\hat{\boldsymbol{B}}\, \hat{\boldsymbol{B}}^\dagger = -\,|\hat{\boldsymbol{B}}|^2 = -\,1 \qquad \text{ainsi} \qquad \hat{\boldsymbol{B}} = i \tag{12.86}$$

- **Phaseur :** exponentielle du bivecteur $\pm\hat{\boldsymbol{B}}\varphi$ dans le plan de rotation

$$e^{\pm\hat{\boldsymbol{B}}\varphi} = \sum_{k=0}^{\infty} \frac{(\pm 1)^k \hat{\boldsymbol{B}}^k \varphi^k}{k!} = \sum_{k=0}^{\infty} \frac{\hat{\boldsymbol{B}}^{2k} \varphi^{2k}}{(2k)!} \pm \sum_{k=0}^{\infty} \frac{\hat{\boldsymbol{B}}^{2k+1} \varphi^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \varphi^{2k}}{(2k)!} \pm \sum_{k=0}^{\infty} \frac{(-1)^k \varphi^{2k+1}}{(2k+1)!} \hat{\boldsymbol{B}} \tag{12.87}$$

- **Formule d'Euler :**

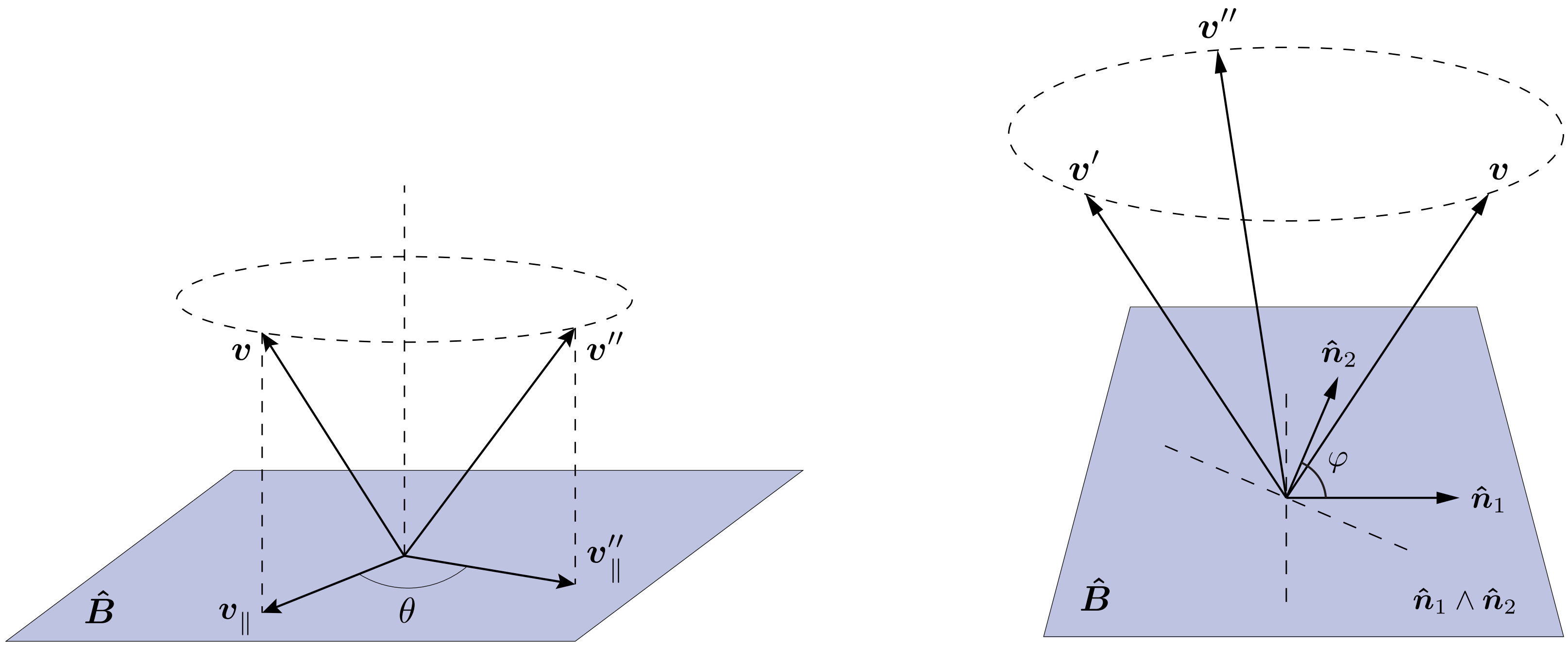
$$\tag{12.88}$$

- **Rotor :** et renversement (12.88) dans (12.85)

$$R = e^{-\hat{\boldsymbol{B}}\varphi} \qquad \text{et} \qquad R^{\dagger} = e^{\hat{\boldsymbol{B}}\varphi} \tag{12.89}$$

- **Rotation :** $\boldsymbol{v} \mapsto \boldsymbol{v}''$

$$\tag{12.90}$$



- **Angles :** (exercice)

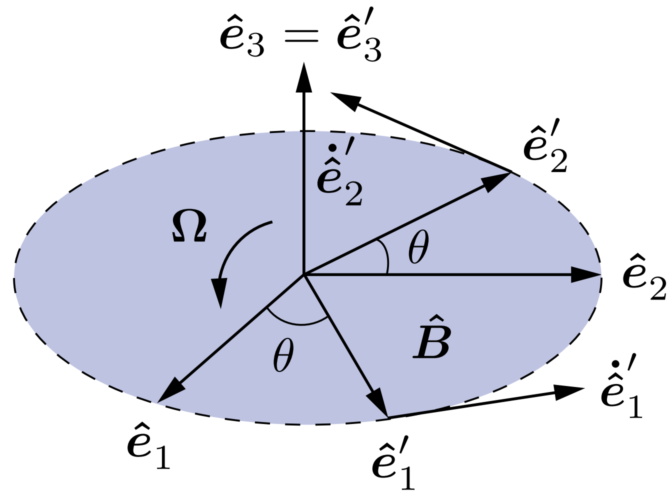
$$\theta = 2\varphi \tag{12.108}$$

- **Rotation :** application linéaire : vecteur $\boldsymbol{v} \ \mapsto \ \boldsymbol{v}''$

(12.109)

- **Rotation :** application linéaire : bivecteur $\boldsymbol{A} \ \mapsto \ \boldsymbol{A}''$ (exercice)

(12.114)



- **Rotation :** $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\} \ \mapsto \ \{\hat{e}_1', \hat{e}_2', \hat{e}_3'\}$

$$\tag{12.115}$$

- **Rotor :** et renversement : bivecteur unitaire $\hat{B}$ constant dans le plan

$$R = e^{-\hat{B}\theta/2} \qquad \text{et} \qquad R^{\dagger} = e^{\hat{B}\theta/2} \tag{12.110}$$

- **Evolution temporelle :** (12.115) et (12.82) où $R^{\dagger}R = 1$ et $\hat{e}_i =$ **cste**

$$\dot{\hat{e}}_i' = \dot{R}\,\hat{e}_i\,R^{\dagger} + R\,\hat{e}_i\,\dot{R}^{\dagger} = \dot{R}\,R^{\dagger}R\,\hat{e}_i\,R^{\dagger} + R\,\hat{e}_i\,R^{\dagger}R\,\dot{R}^{\dagger}$$

$$= \dot{R}\,R^{\dagger}\,\hat{e}_i' + \hat{e}_i'\,R\,\dot{R}^{\dagger} \tag{12.117}$$

- **Rotor :** isométrie où $R^{\dagger} = R^{-1}$

$$R\,R^{\dagger} = 1 \qquad \text{ainsi} \qquad \dot{R}\,R^{\dagger} + R\,\dot{R}^{\dagger} = 0 \tag{12.118}$$

- **Evolution temporelle :** rotor

$$\dot{R}\,R^{\dagger} = -\,\frac{\dot{\theta}}{2}\,\hat{\boldsymbol{B}}\,e^{-\,\hat{\boldsymbol{B}}\theta/2}\,e^{\hat{\boldsymbol{B}}\theta/2} = -\,\frac{\dot{\theta}}{2}\,\hat{\boldsymbol{B}}$$

$$R\,\dot{R}^{\dagger} = e^{-\,\hat{\boldsymbol{B}}\theta/2}\,e^{\hat{\boldsymbol{B}}\theta/2}\,\frac{\dot{\theta}}{2}\,\hat{\boldsymbol{B}} = \frac{\dot{\theta}}{2}\,\hat{\boldsymbol{B}} \tag{12.120}$$

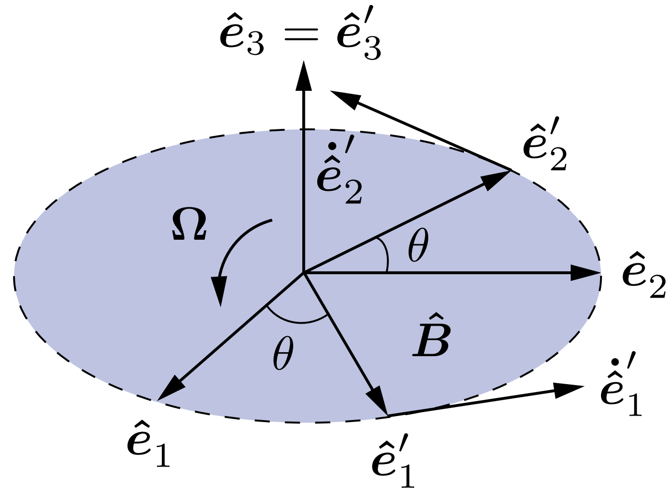
- **Bivecteur vitesse angulaire :** dans le plan

(12.121)

- **Evolution temporelle :** rotor et renversement $(12.118) - (12.121)$

$$\dot{R}\,R^{\dagger} = -\,\frac{1}{2}\,\boldsymbol{\Omega} \qquad \text{ainsi} \qquad \dot{R} = -\,\frac{1}{2}\,\boldsymbol{\Omega}\,R$$

$$R\,\dot{R}^{\dagger} = \frac{1}{2}\,\boldsymbol{\Omega} \qquad \text{ainsi} \qquad \dot{R}^{\dagger} = \frac{1}{2}\,R^{\dagger}\,\boldsymbol{\Omega} \tag{12.122}$$



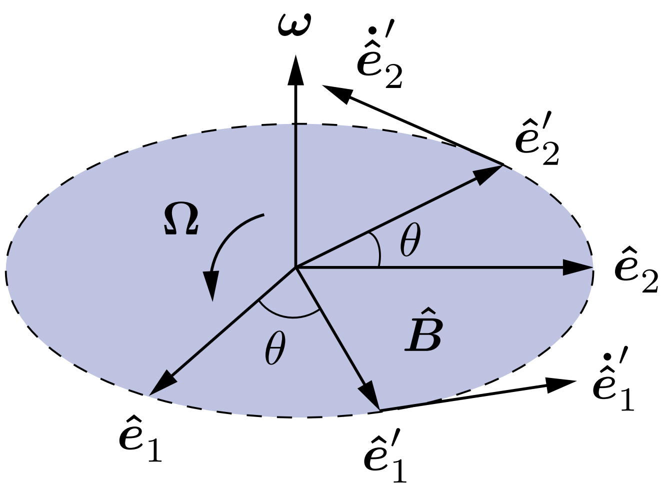
- **Rotor et renversement :** angle initial $\theta(0) = 0$ ainsi $R(0) = R^{\dagger}(0) = 1$

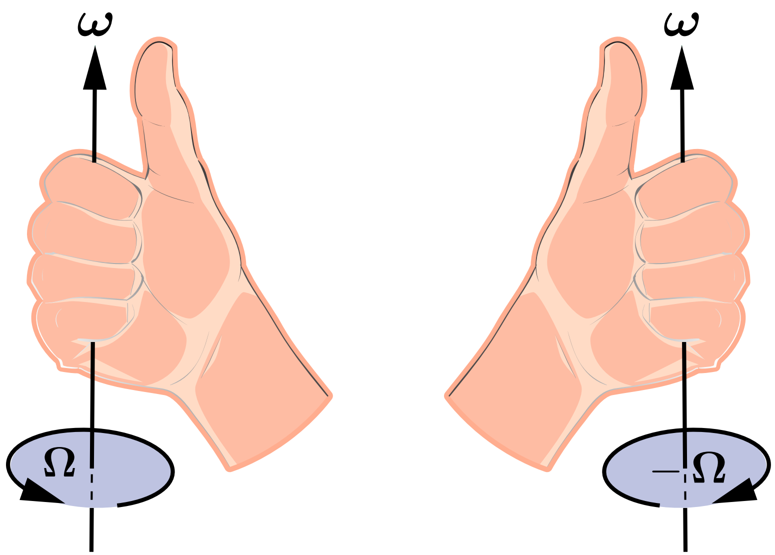
$$R(t) = \exp\left(-\frac{1}{2}\int_0^t \boldsymbol{\Omega}(t')\,dt'\right) = \exp\left(-\hat{\boldsymbol{B}}\,\frac{\theta(t)}{2}\right)$$

$$R^{\dagger}(t) = \exp\left(\frac{1}{2}\int_0^t \boldsymbol{\Omega}(t')\,dt'\right) = \exp\left(\hat{\boldsymbol{B}}\,\frac{\theta(t)}{2}\right) \qquad (12.125)$$

- **Formule de Poisson :** algèbre géométrique (12.122) dans (12.117)

$$(12.127)$$





- **Vitesse angulaire :** pseudovecteur et bivecteur

$$\boldsymbol{\Omega} \cdot \boldsymbol{\omega} = 0 \qquad \text{et} \qquad |\boldsymbol{\Omega}| = |\boldsymbol{\omega}| \tag{12.130}$$

- **Dualité :** règles de la main droite et de la main gauche

$$\boldsymbol{\Omega}^* = \boldsymbol{\omega} \qquad \text{et} \qquad \boldsymbol{\omega}^* = -\,\boldsymbol{\Omega} \tag{12.129}$$

- **Identité algébrique :** (12.129)

$$\hat{\boldsymbol{e}}_i' \cdot \boldsymbol{\Omega} = -\,\hat{\boldsymbol{e}}_i' \cdot \boldsymbol{\omega}^* = -\,\left(\hat{\boldsymbol{e}}_i' \wedge \boldsymbol{\omega}\right)^* = -\,\hat{\boldsymbol{e}}_i' \times \boldsymbol{\omega} = \boldsymbol{\omega} \times \hat{\boldsymbol{e}}_i' \tag{12.132}$$

- **Formule de Poisson :** espace vectoriel $\mathbb{R}^3$ (12.132) dans (12.127)

$$\tag{12.134}$$